

Second Best Efficiency and the English Auction*

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Abstract

We study the second best in a single unit sale to two bidders. This is the allocation that maximizes the expected social surplus subject to the bidders' incentive compatible constraints when the first best is not implementable. We prove that Maskin's (1992) result that any first best allocation that is deterministic and monotone can be implemented with the English auction carries over to the second best.

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1 Introduction

Suppose that an oil tract is put up for sale between two wildcatters. The first one, the incumbent, has a high marginal cost and a low fixed cost, whereas the second one, the entrant, has a low marginal cost and a high fixed cost. In this case, it may be efficient to allocate the good to the incumbent if there is little oil and to the entrant if there is much oil. However, Maskin (1992) has shown that this allocation, i.e. the *first best*, is not implementable¹ when the amount of oil is private information of the incumbent. A similar problem may arise in an auction with an insider and an outsider. What is then the socially optimal allocation subject to implementability, i.e. the *second best*? Can it be implemented with a “realistic” mechanism?

Maskin (1992) has shown that for the case of only two bidders, and if value functions verify a mild monotonicity condition, any first best allocation that is deterministic and monotone^{2,3} can be implemented with the English auction. We show that this result also holds true for the second best. In fact, we derive this result from the more general claim that the English auction implements the deterministic and monotone allocation that gives the greatest expected surplus when there are only two bidders.

Restricting to monotone allocations is natural if one is interested in the English auction since only such allocations can be implemented as a monotone equilibrium of the English auction. The restriction to deterministic allocations only eliminates equilibria in mixed strategies and equilibria with ties. It is unclear how these could improve efficiency. Besides, both restrictions are without loss of generality for the natural case in which values are additively separable and private types stochastically

¹In this paper, we mean by implementable that the allocation is the equilibrium outcome to some game. This notion of implementability is also called achievability. Note that it differs from the concept of full implementation. This latter concept requires that the allocation is the unique equilibrium outcome to some game.

²For a formal definition of deterministic and monotone allocations see our Section 4. Informally, an allocation is deterministic if it does not involve lotteries and it is monotone if the ex post probability of allocating the good to a bidder does not go down as we increase her type keeping constant the rival’s type.

³This result is stated in terms of an almost equivalent single crossing condition.

independent. Our motivating examples verify these assumptions.

To grasp the intuition of our result consider the wildcatter example. Since the incumbent knows the amount of oil, she knows her value. Hence, she has a unique weakly dominant strategy as in a private value auction: to stay active until her value is reached. If the incumbent plays this strategy, the entrant's payoff when winning is equal to the difference between her value and the incumbent's, i.e. it is equal to the change in social surplus. Thus, the entrant's best reply maximizes not only her expected profits but also social welfare, i.e. the private interest of the entrant is aligned with the social one. In our analysis, we use the local optimality conditions of the second best problem to show how an adaptation of this intuition applies more generally.

The above example also illustrates that the strategic analysis of the English auction when the first best is not implementable is more complex than otherwise. Since the entrant's value is greater than the incumbent's if and only if the latter is large enough, the entrant makes a loss if she wins at a low price but a profit if she wins at a high price. Her best response must trade off these expected losses and gains. It may be possible that the entrant finds it profitable to remain in the auction at prices at which she makes a loss when the incumbent quits, i.e. there may be ex post regret in equilibrium.

Ex post regret implies that the equilibrium is not an ex post equilibrium. This is a common feature of second best efficient equilibria of the English auction. It is easy to see why. If the first best is not implementable, the second best allocation maximizes total expected surplus trading-off inefficient allocations with the weights given by the bidders' beliefs. In general, this means that the second best allocation varies with the bidders' beliefs and, consequently, any equilibrium that implements it must also vary with them. Generally, this is incompatible with ex post equilibria. An alternative is a more sophisticated auction mechanism whose rules vary with the bidders' beliefs. This, however, is less appealing as emphasized by Wilson's critique.

The rest of the paper is organized as follows. The related literature is in Section 2. We define the formal set-up in Section 3. Section 4 contains several definitions and Section 5 some motivating examples in which the first best is not implementable. We study the English auction in Section 6. Section 7 concludes and the Appendix contains

the most involved proofs.

2 Related Literature

Most of the papers that study the set of auction mechanisms that maximize the expected social surplus subject to the buyers' incentive compatibility constraints differ from ours in that they assume conditions that guarantee that the incentive compatibility constraints are not binding. This is for instance the case of Vickrey (1961), Krishna and Perry (1998), and Williams (1999), and most of the analysis of Maskin (1992, 2000), and Dasgupta and Maskin (2000).

Maskin (1992, 2000), Dasgupta and Maskin (2000), Eso and Maskin (2000), and Jehiel and Moldovanu (2001) also consider the case in which the first best is not implementable. Their results, however, hinge on the assumption that bidders have multidimensional private information. They argue that in this case an implementable allocation cannot depend on the type beyond a particular one-dimensional reduction. The first best is usually not implementable because it requires conditioning on more information than this one-dimensional reduction. Eso and Maskin (2000) define in this set-up the constraint efficient allocation. This is the allocation that maximizes expected social surplus when we can only condition the allocation on the former one-dimensional reduction.⁴

Another related branch of the literature, Maskin (1992), Krishna (2003), Birulin and Izmalkov (2009), Izmalkov (2003), and Dubra, Echenique, and Manelli (2009), analyzes whether there is an equilibrium of the English auction that allocates the good efficiently when the efficient allocation is implementable.

On the technical side, our analysis of the case of independent types and additively separable value functions is related to the ironing technique introduced by Mussa and

⁴Although we assume a one-dimensional type space, we show in the working paper version of this paper, Hernando-Veciana and Michelucci (2008), that some of our results may be used in the efficiency analysis based on the one-dimensional reduced types. In a more general version of our example in Section 5.2, it is generally the case that the one-dimensional reduction does not verify the conditions required for implementability of the constraint efficient allocation.

Rosen (1978) and Myerson (1981). In a recent paper, Boone and Goeree (2009) have used the ironing technique in an environment closely related to our motivating example in Section 5.2. Their focus, as in Myerson (1981), is on revenue maximization rather than on efficiency.

The problem of second best efficiency has also received attention in the context of two parties that bargain with asymmetric information, see Myerson and Satterthwaite (1983). The difference is that in their set-up withdrawing the individual rationality constraints always makes the first best implementable, whereas this is not the case in our set-up. In fact, we consider the usual auction environment in which the individual rationality constraints can be trivially met and it is only the incentive compatibility constraints that may be binding.

3 The Model

One unit of an indivisible good is put up for sale to a set of two bidders $\{1, 2\}$. Let $s = (s_1, s_2)$ be a vector that it is equal to the realization of a random variable with distribution F and with a strictly positive bounded density f in a bounded support $S \equiv S_1 \times S_2 \subset \mathbb{R}^2$. We denote each marginal distributions of F (on the sets S_i 's) by F_i and its density by f_i . Bidder i observes privately s_i and gets a von Neumann-Morgenstern utility $v_i(s) - P$ if she gets the good for sale at price P , and utility $-P$ if Bidder j , $j \neq i$, gets the good and i pays a price P . We assume that v_i is non-negative, bounded, measurable and strictly increasing in s_i , for any i .⁵

Let an *allocation* be a measurable function $p : S \rightarrow [0, 1]^2$, such that $p_1(s) + p_2(s) = 1$ for any $s \in S$, where $p_i(s)$ denotes the probability that the good is allocated to i when the vector of types is $s \in S$. Note that we do not allow for the possibility that the good remains unsold. This is a common assumption in the papers that study the efficiency of the English auction, for instance Maskin (2000), Krishna (2003), Birulin and Izmalkov

⁵We say that a function $g : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is increasing if for any $x, y \in D$, $x > y$ implies $g(x) \geq g(y)$, and strictly increasing if $x > y$ implies $g(x) > g(y)$.

(2009), and Dubra, Echenique, and Manelli (2009).^{6,7}

We also make another common assumption, that $v_i(s)$ is increasing. It is well-known that an assumption of this sort is necessary for the English auction to implement the first best. That a similar condition is also necessary for the second best is a consequence of the fact that the strategies that implement it share with the strategies that implement the first best that each bidder bids her value conditional on his type and on the other bidder's type being pivotal for the allocation.⁸

4 Definitions

Definition: We say that an allocation p is *first best* when $\forall s \in S, p_i(s) > 0$ only if:

$$v_i(s) \geq v_j(s), \quad j \neq i.$$

We are interested in the set of allocations that can be implemented. By the revelation principle, there is no loss of generality in restricting to direct mechanisms. A *direct mechanism* is a pair of measurable functions (p, x) , where p is an allocation and $x : S \rightarrow \mathbb{R}^2$ a payment function. In the direct mechanism (p, x) , each bidder announces a type, and $p_i(s)$ denotes the probability that i gets the good and $x_i(s)$ her payment to the auctioneer when the vector of announced types is $s \in S$.

The expected utility of Bidder i with type s_i who reports s'_i when the other bidder

⁶This assumption is without loss of generality if the seller's value is sufficiently small relative to the buyer's valuations. Indeed, our assumption that retaining the good is not an option for the seller can be interpreted as if retaining the good for the seller has minus infinity value. To some extent, this may be the case for some government auctions where not selling is not option.

⁷In fact, there is usually no combination of entry fees and reserve prices that enables the English auction to implement the efficient allocation when this allocation requires that the good remains unsold for some realizations of the bidders' types.

⁸We cannot directly relax our assumption as in Krishna (2003), Birulin and Izmalkov (2009), or Dubra, Echenique, and Manelli (2009). The reason is that their alternative assumptions require assuming continuity of the value functions, which conflicts with our motivating example in Section 5.2.

reports truthfully is equal to:⁹

$$U_i(s_i, s'_i) \equiv \int_{S_j} (v_i(s_i, s_j)p_i(s'_i, s_j) - x_i(s'_i, s_j)) f_j(s_j|s_i) ds_j,$$

where $j \neq i$ and $f_j(s_j|s_i)$ denotes the density of the distribution of s_j conditional on the realization of Bidder i 's type being equal to s_i .

Thus, we say that an allocation p is *implementable* if there exists a direct mechanism (p, x) that satisfies the following Bayesian incentive compatible constraint:¹⁰

$$U_i(s_i, s_i) = \sup_{s'_i \in S_i} \{U_i(s_i, s'_i)\},$$

for all $s_i \in S_i$ and $i \in \{1, 2\}$.

Requiring Bayesian incentive compatibility constraints is consistent with the fact that we consider Bayesian Nash equilibria of the English auction. This departs from the alternative approach used in the related literature that employs ex post equilibrium and ex post incentive compatible constraints. This departure, as explained in the Introduction, is motivated by our focus on an auction whose details are not fine tuned to the bidders' beliefs.

Definition: We say that an implementable allocation p is *second best efficient*, if it maximizes the expected social surplus:

$$\int_S \sum_{i=1,2} v_i(s)p_i(s)f(s) ds,$$

among the implementable allocations.

Since we assume preferences quasilinear in money, second best efficiency is equivalent to Holmström and Myerson's (1983) concept of ex ante incentive efficiency.

Certainly, the set of second best allocations includes the first best allocation when the efficient allocation is implementable.

Our interest in the English auction, see our discussion in the Introduction, justifies to concentrate on second best allocations that are deterministic and monotone in the

⁹With some abuse of notation, we denote by $p_i(s_i, s_j)$, $x_i(s_i, s_j)$ and $v_i(s_i, s_j)$ the functions p_i , x_i and v_i , respectively, evaluated at (s_i, s_j) if $i < j$ and at (s_j, s_i) otherwise.

¹⁰We do not impose individual rationality constraints because they are trivially satisfied in our set-up.

following sense: an allocation is *deterministic* if it takes values in the set $\{0, 1\}^2$, and it is *monotone* if $p_i(s)$ is increasing in s_i for any i . We also say that an allocation is *monotone in i* when $p_i(s)$ is increasing in s_i . This last definition is useful to state some of our results in particular Proposition 2.

Monotone allocations have been extensively studied by the related literature because monotonicity is equivalent to ex post implementability, see Chung and Ely (2002). In this sense, we can see our restriction to monotone allocations as a refinement of Bayesian implementation. Besides, we show in the next proposition that our restriction is without loss of generality under some reasonable assumptions that, for instance, are verified by our motivating examples in Section 5.

Proposition 1. *There exists a deterministic and monotone allocation that it is second best efficient when bidders' types are stochastically independent and the value functions are additively separable.*¹¹

See the proof in the Appendix.

The proof of Proposition 1 contains a characterization of the second best efficient allocations that we use in the analysis of Example 1 in Section 6.

5 Motivating Examples

In this section, we discuss two simple examples that illustrate realistic set-ups in which the first best is typically not implementable. The first set-up is the sale of a license to operate in a market in which one bidder, the incumbent, has private information about the market size and another bidder, the entrant, has a lower marginal cost than the incumbent but a higher fixed cost. This model extends the intuitions of the wildcatters' example in the Introduction. The second set-up is the auction of an item with private and common values in which the common value is private information of only one bidder, the insider. The two examples verify Myerson's (1981) assumptions, and thus his Lemma 2 implies that the first best allocation is not implementable if

¹¹This is $v_i(s) = v_i^i(s_i) + v_i^j(s_j)$ for some $v_i^i : S_i \rightarrow \mathbb{R}$ and $v_i^j : S_j \rightarrow \mathbb{R}$.

the corresponding interim probabilities of getting the object are not increasing in the bidder's type.

5.1 An Auction with an Incumbent

The object for sale is a license to supply in exclusivity a market with a demand function $Q(P) = s_1(1-P)$. Bidders are firms that can supply the market. Firm 1, *the incumbent*, has zero set-up costs and a constant marginal cost c_1 . Firm 2, *the entrant*, has a constant marginal cost c and an idiosyncratic set-up cost that we denote by $-s_2$, thus a higher s_2 means a lower fixed cost. Each s_i is equal to the realization of an independent random variable with distribution function F_i and a density. Its realization is Bidder i 's private information. All the other elements of the model are common knowledge.

The profits of the incumbent with type s_1 if awarded the license are equal to $s_1 \frac{(1-c_1)^2}{4}$. Similarly, the profits of an entrant with type s_2 are equal to $s_1 \frac{(1-c)^2}{4} + s_2$. Thus, it is first best to give the license to the incumbent if and only if $s_1 \frac{(1-c_1)^2}{4} \geq s_1 \frac{(1-c)^2}{4} + s_2$. We also assume that $s_2 < 0$ in all the support and $c < c_1 < 1$, so that, as in the example of the Introduction, the incumbent has the lowest fixed cost, but the highest marginal cost. This assumption implies that it is first best to allocate to Bidder 1 if and only if s_1 , i.e. the market size, is sufficiently small. It is easy to see that under some appropriate assumptions on the support of the distributions F_i , the first best allocation is not implementable because Bidder 1's interim probability of getting the object strictly decreases at some point when we increase Bidder 1's type.

5.2 An Auction with an Insider

The object for sale is a painting that may be from a well-known (and priced) artist. Bidder i puts a value on the painting of $\tau_i + \rho$ if the painting is original and a value of τ_i , otherwise. We assume that each τ_i is equal to an independent draw of a random variable with a distribution function G_i and a density in the support $[\underline{t}, \bar{t}]$. We assume that τ_i is private information of Bidder i . Bidder 1, *the insider*, is an expert art dealer that knows whether the painting is original. Bidder 2, *the outsider*, only knows the ex ante probability that the picture is original, $\alpha \in (0, 1)$.

We also assume that $\bar{t} < \rho + \underline{t}$. This assumption means that Bidder 1's multidimensional type can be mapped into a one dimensional type, $s_1 = \tau_1 + \rho$, without losing information.¹² Note that $s_1 \geq \rho + \underline{t}$ indicates that the painting is original. Bidder 2's type s_2 is equal to τ_2 . Thus, the outsider's value is equal to $s_2 + \rho \mathbf{1}(s_1 \geq \rho + \underline{t})$.¹³ For any vector of the outsiders' types, it is efficient to allocate to the insider if $s_1 = \bar{t}$, while it is not if $s_1 = \underline{t} + \rho$. This means that the first best allocation is not implementable because Bidder 1's interim probability of getting the object strictly decreases when s_1 increases from \bar{t} to $\underline{t} + \rho$.

6 The English Auction

In this section, we analyze the English auction described by Milgrom and Weber (1982) and show that any second best allocation that it is deterministic and monotone can be induced as the outcome of an equilibrium of the English auction, i.e. can be implemented with the English auction. Note that since we assume that there are only two bidders, the English auction is strategically equivalent to a second price auction.

In what follows, we assume $S_i = [0, 1]$ to simplify the notation. A deterministic and monotone allocation in j is characterized a.e. with a function of pivotal types $\psi_j : [0, 1] \rightarrow [0, 1]$ that maps each type of Bidder i into the corresponding type of Bidder j that is pivotal for the allocation. The allocations that maximize the expected social surplus amongst the allocations that are deterministic and monotone in j are characterized by the functions of pivotal types ψ_j that solve:

$$\max_{\psi_j} \int_0^1 \int_0^{\psi_j(s_i)} (v_i(s) - v_j(s)) f(s) ds_j ds_i. \quad (1)$$

This problem maximizes the net increase in expected social surplus when adopting the allocation characterized by ψ_j rather than an allocation that always allocates to Bidder j .

¹²Our results can be extended to the case $\rho + \underline{t} \leq \bar{t}$, but it requires a framework with multidimensional types. See also Footnote 4.

¹³ $\mathbf{1}(X)$ is an indicator function that takes value 1 when the condition X is verified and zero otherwise.

We start with the case in which one bidder knows her value and hence has a unique weakly dominant strategy, to bid her value.

Proposition 2. *Suppose $v_1(s)$ is constant in s_2 . The undominated equilibria of the English auction implement a.e. the allocations that solve Equation (1) for $j = 1$ and $i = 2$. Consequently, any second best allocation that is deterministic and monotone in 1 can be implemented a.e. with the English auction.*

Proof. We assume that Bidder 1 plays her unique weakly dominant strategy, to bid her value, and study Bidder 2's best response. A strategy of Bidder 2 is a function that maps each type of Bidder 2 into a bid. Since Bidder 1's strategy is strictly increasing, Bidder 2's strategy determines a function of pivotal types that maps each type of Bidder 2 into the maximum type of Bidder 1 for which Bidder 2 wins the auction. Thus, the expected utility that Bidder 2 gets with a strategy that has an associated function of pivotal types $\psi_1 : [0, 1] \rightarrow [0, 1]$ is equal to:

$$\int_0^1 \int_0^{\psi_1(s_2)} (v_2(s) - v_1(s)) f(s) ds_1 ds_2,$$

since whenever Bidder 2 wins the auction, she pays Bidder 1's bid, i.e. Bidder 1's value. A best response of Bidder 2 picks a function ψ_1 that maximizes this expression as desired. ■

Intuitively, since Bidder 1 bids her true value, Bidder 2's utility when she wins is equal to the increase in expected social surplus from changing the allocation from Bidder 1 to Bidder 2. Thus, Bidder 2's incentives are aligned with the social incentives. Since Bidder 1 plays a strictly increasing bid function, then a pure strategy of Bidder 2 can only pick an allocation that is deterministic and monotone in 1.

For the more general case, we focus on deterministic and monotone allocations. Any such allocation is characterized a.e. by an increasing function of pivotal types. The allocations that maximize the expected social surplus amongst the deterministic and monotone allocations are characterized by the functions of pivotal types that solves Equation (1) subject to the constraint that the function of pivotal types is increasing. For any such allocation p^* , we define the bid functions $b_1^*(s_1) \equiv v_1(s_1, \psi_2^*(s_1))$ and

$b_2^*(s_2) \equiv v_2(\psi_1^*(s_2), s_2)$, where ψ_1^* and ψ_2^* are the functions of pivotal types that describe p^* a.e.

Proposition 3. *Suppose f is affiliated.¹⁴ Then, (b_1^*, b_2^*) is an equilibrium of the English auction that implements p^* a.e. Consequently, any second best allocation that is deterministic and monotone can be implemented a.e. with the English auction.*

See the proof in the Appendix.

The main argument of the proof¹⁵ uses that the local optimality conditions of the problem that ψ_j^* solves imply that $v_i(s_i, \psi_j^*(s_i)) \geq v_j(s_i, \psi_j^*(s_i))$ at any point s_i in which ψ_j^* is strictly increasing either to the left or to the right.¹⁶ This implies that $b_i^*(s'_i) > b_j^*(s'_j)$ for any (s'_i, s'_j) in the interior of the set of types for which p^* allocates to Bidder i . This is because there always exists a point s_i such that $(s'_i, \psi_j^*(s'_i)) > (s_i, \psi_j^*(s_i)) > (\psi_i^*(s'_j), s'_j)$ and at which ψ_j^* is strictly increasing either to the left or to the right. Finally, to understand why (b_1^*, b_2^*) is an equilibrium note that when Bidder j plays b_j^* , Bidder i 's expected utility when she wins is equal to $v_i(s) - b_j^*(s_j)$. This is greater (less) than the social incentives to allocate to Bidder i rather than to Bidder j , $v_i(s) - v_j(s)$, if $s_i \geq \psi_i^*(s_j)$ (resp. $s_i < \psi_i^*(s_j)$). Thus, Bidder i 's incentives to get the good are greater (less) than the social incentives to allocate to Bidder i if the second best p^* allocates the good to Bidder i (resp. Bidder j). This explains why Bidder i does not have incentives to deviate since p^* maximizes expected social surplus subject to monotonicity of the allocation and affiliation guarantees that Bidder i does not lose by restricting to an increasing best response.

To illustrate the equilibrium, consider the following example that has all the qualitative features of the model in Section 5.2.

Example 1. $v_1(s) = 2s_1$ and $v_2(s) = s_2 + \mathbf{1}(s_1 \geq 1/2)$, where bidders' private types are drawn independently according to a distribution function uniform in the support $[0, 1]$.

¹⁴See Milgrom and Weber (1982) for a definition of affiliation.

¹⁵For illustrative purposes, we assume in our intuitive description that the functions involved are continuous.

¹⁶We say that an increasing function $\psi : D \subset \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing to the left (resp., to the right) at a point $s \in D$, when $\psi(\tilde{s}) < \psi(s)$ for any $\tilde{s} < s$ in D (resp. $\psi(\tilde{s}) > \psi(s)$ for any $\tilde{s} > s$ in D).

In this example, the second best efficient allocation is characterized¹⁷ by a function of pivotal types $\psi_1^*(s_2) = \frac{1}{2}(s_2 + \mathbf{1}(s_2 \geq 1/2))$, and our proposed equilibrium strategies are: $b_1^*(s_1) = 2s_1$ and $b_2^*(s_2) = s_2 + \mathbf{1}(s_2 \geq 1/2)$. We plot both bid functions in Figure 1.

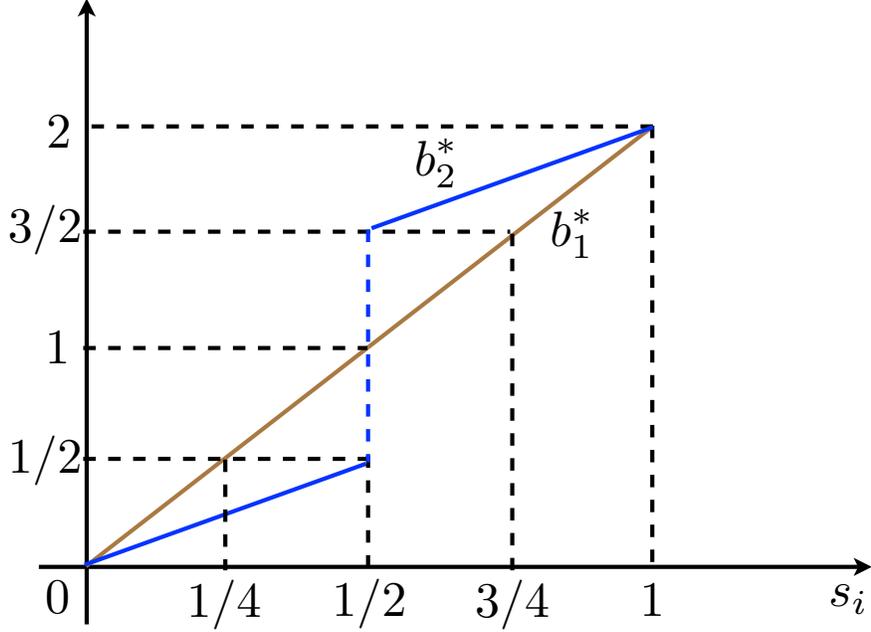


Figure 1: The bid functions (b_1^*, b_2^*) for Example 1.

Bidder 1 submits her unique weakly dominant bid, her value, whereas Bidder 2 bids her value conditional on the event that she ties with Bidder 1. Bidder 2 has a discontinuity at $s_2 = 1/2$, it jumps from $1/2$ to $3/2$. It is easy to see that when Bidder 2 has a type of $1/2$ and wins at a price p between $1/2$ and 1 , she incurs in a loss, whereas when the price is between 1 and $3/2$, she gets a profit. Losses and profits compensate so that Bidder 2 with type $1/2$ is indifferent between bidding $1/2$ and bidding $3/2$.

Note that the structure of this equilibrium is more involved than the more standard

¹⁷To see why, we refer to the characterization of the second best in the proof of Proposition 1. To apply this characterization note that $t_1(s_1) = 2s_1 - \mathbf{1}(s_1 \geq 1/2)$ and $t_2(s_2) = s_2$ in the example, and hence, $\hat{t}_1(s_1) = 2s_1$ if $s_1 \in [0, 1/4]$, $\hat{t}_1(s_1) = 1/2$ if $s_1 \in (1/4, 3/4)$, $\hat{t}_1(s_1) = 2s_1 - 1$ if $s_1 \in [3/4, 1]$; and $\hat{t}_2(s_2) = s_2$.

case in which the first best is implementable. The difference is that, as we explained in the previous paragraph, Bidder 2 may win the auction at prices at which she makes a loss, i.e. there may be ex-post regret. This is a common feature of the equilibria under the assumptions of Proposition 2 when the first best allocates to Bidder 1 and the second best to Bidder 2. To see why, note that the first best allocates to the bidder with larger value and Bidder 2 pays a price equal to Bidder 1's value when she wins in equilibrium. This result, however, does not generalize to the case in which Bidder 1's value function is not constant in s_2 . There are equilibria that implement the second best that display ex post regret and equilibria that implement the second best that do not display ex post regret.¹⁸

7 Conclusions

In this paper we have discussed the efficiency properties of the English auction when the first best is not implementable and there are two bidders. The English auction implements the second best allocation at least for the relevant case in which the second best is deterministic and monotone. Implementing the second best outside this case seems much harder as it may require the use of either non-monotonic strategies or other mechanisms with more sophisticated strategy spaces.

Our result does not generalize to the case of more than two bidders. This was well-known in the case in which the first best is implementable, see for instance Maskin (1992). The conditions provided by Krishna (2003), Birulin and Izmalkov (2009), and Dubra, Echenique, and Manelli (2009) to guarantee that the first best can be implemented with the English auction with more than two bidders do not easily generalize to the second best, as we show in the working paper. One additional difficulty here is that the equilibrium of the English auction that implements the second best with two bidders typically implies ex post regret. This means that the modifications of the

¹⁸In general, multiplicity of equilibria is more a concern here than in the usual framework where the single crossing holds. It may be shown that in the example $v_i(s) = s_i + 2s_j$ there are two asymmetric equilibria that implement the second best while the symmetric equilibrium is not second best efficient. Besides, there is no natural refinement that singles out an equilibrium.

English auction proposed by Perry and Reny (2002, 2005) and Izmalkov (2003) to solve the problems of the first best does not work for the second best.

We think that a profitable venue of future research is how to modify the English auction for the case of more than two bidders to recover second best efficiency, at least for the natural examples we have considered here. We find this project specially appealing given the connection between the English auction and the Vickrey auction.

Appendix: Proofs

Proof of Proposition 1

Proof. Recall the functions v_i^i and v_i^j in Footnote 11 and denote in this proof $t_i(s_i) \equiv v_i^i(s_i) - v_i^j(s_i)$ and $q_j(s_j) \equiv v_j^i(s_j)$, $j \neq i$. To simplify the notation, we normalize the marginal distributions F_i 's to be uniform in the interval $[0, 1]$.¹⁹

Let $T_i(s_i) \equiv \int_0^{s_i} t_i(\tilde{s}_i) d\tilde{s}_i$ for all $i \in n$ and $s_i \in [0, 1]$, and let $\hat{T}_i(s_i) : [0, 1] \rightarrow \mathbb{R}$ be the convex hull of the function T_i (i.e. the highest convex function on $[0, 1]$ such that $\hat{T}_i(s_i) \leq T_i(s_i)$ for all $s_i \in [0, 1]$.)²⁰

As a convex function \hat{T}_i is differentiable except at countably many points, and its derivative is an increasing function. We define $\hat{t}_i : [0, 1] \rightarrow \mathbb{R}$ to be the differential of \hat{T}_i completed by right-continuity in the interior and by continuity at the boundaries.

The second best maximizes:

$$\int_{[0,1]^2} \sum_{i=1,2} \left(t_i(s_i) + \sum_{j=1,2} q_j(s_j) \right) p_i(s) ds, \quad (2)$$

¹⁹This normalization was already noted by Lehmann (1988). We can always construct it by defining a new vector of signals $\tilde{s}_i \equiv F_i(s_i)$ and value functions $\tilde{v}_i(\tilde{s}) \equiv \tilde{t}_i(\tilde{s}_i) + \sum_{i=1,2} \tilde{q}_j(\tilde{s}_j)$ where $\tilde{t}_i(\tilde{s}_i) \equiv t_i(F_i^{-1}(\tilde{s}_i))$, $\tilde{q}_j(\tilde{s}_j) \equiv q_j(F_j^{-1}(\tilde{s}_j))$, for $F^{-1}(z) \equiv \min\{s_i \in [\underline{s}, \bar{s}] : F(s) \geq z\}$. To see why, the marginal distribution of each \tilde{s}_i is uniform on $[0, 1]$ note that the probability of $\{\tilde{s}_i \leq z\}$ for $z \in [0, 1]$ is equal to the probability of $\{F_i(s_i) \leq z\}$, which is equal to the probability of $\{s_i \leq F_i^{-1}(z)\}$ and thus, it is equal to $F_i(F_i^{-1}(z)) = z$.

²⁰For a formal definition see Myerson (1981).

subject to p implementable. The objective function is equal to:²¹

$$\int_{[0,1]^2} \sum_{i=1,2} \left(\hat{t}_i(s_i) + \sum_{j=1,2} q_j(s_j) \right) p_i(s) ds + \sum_{i=1,2} \int_0^1 \left(\hat{T}_i(s_i) - T_i(s_i) \right) dQ_i(s_i, p), \quad (3)$$

where:

$$Q_i(s_i, p) \equiv \int_0^1 p_i(s_i, s_j) ds_j.$$

To see why, note that:

$$\begin{aligned} \int_{[0,1]^2} (t_i(s_i) - \hat{t}_i(s_i)) p_i(s) ds &= \int_0^1 (t_i(s_i) - \hat{t}_i(s_i)) Q_i(s_i, p) ds_i \\ &= \int_0^1 Q_i(s_i, p) dT_i(s_i) - \int_0^1 Q_i(s_i, p) d\hat{T}_i(s_i) \\ &= \int_0^1 \left(\hat{T}_i(s_i) - T_i(s_i) \right) dQ_i(s_i, p), \end{aligned}$$

where we have used integration by parts (see Hewitt (1960)) and the fact that $\hat{T}_i(0) = T_i(0)$ and $\hat{T}_i(1) = T_i(1)$, see Section 6 in Myerson (1981), in the last step.

It is easy to see that an allocation maximizes the first integral in Equation (3) if and only if it satisfies (i) $p_i^*(s) > 0$ only if $\hat{t}_i(s_i) \geq \hat{t}_j(s_j)$, $j \neq i$ a.e. Moreover, since $Q_i(\cdot, p)$ is increasing for any p implementable by Lemma 2 in Myerson (1981), and $\hat{T}_i(s_i) \leq T_i(s_i)$, see Section 6 in Myerson (1981), an implementable allocation maximizes the second integral if and only if it satisfies $Q_i(\cdot, p^*)$ is constant in any open interval in which $\hat{T}_i(s_i) < T_i(s_i)$.

A deterministic and monotone allocation that verifies (i) and (ii) is:

$$p_i(s) = \begin{cases} 1 & \text{if } i = \min\{j : \hat{t}_j(s_j) = \max_l \hat{t}_l(s_l)\}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in \{1, 2\}$. ■

Proof of Proposition 3

The proof follows from four claims that we prove below. The first and third claim are auxiliary results, the second claim is that the strategies (b_1^*, b_2^*) induce the allocation

²¹We denote by $\int_E \varphi(x) dF(x)$ the Lebesgue-Stieljes integral of φ with respect to F in E . In particular, for any implementable allocation p , we denote by $\int_{S_i} \varphi(s_i) dQ_i(s_i, p)$ the Lebesgue-Stieljes integral of φ with respect to $Q_i(\cdot, p)$ in S_i .

p^* , and the fourth that (b_1^*, b_2^*) is an equilibrium.

Claim 1: ψ_j^* verifies that:

$$\lim v_i(s_i^n, \psi_j^*(s_i^n)) \geq \lim v_j(s_i^n, \psi_j^*(s_i^n)),$$

for any decreasing sequence $\{s_i^n\}$ that tends to a point $\hat{s}_i \in [0, 1]$ at which ψ_j^* is strictly increasing either to the left or to the right.

Proof. To prove the claim, it is sufficient to show that for any $\hat{s}'_i \in (\hat{s}_i, 1]$:

$$\int_{\hat{s}_i}^{\hat{s}'_i} (v_i(s_i, \psi_j^*(s_i)) - v_j(s_i, \psi_j^*(s_i))) f(s) ds_i \geq 0.$$

To prove so, note that the optimality of ψ_j^* implies that for any increasing function $\hat{\psi}_j : [0, 1] \rightarrow [0, 1]$:

$$\int_0^1 \int_0^{\psi_j^*(s_i)} (v_i(s) - v_j(s)) f(s) ds_j ds_i \geq \int_0^1 \int_0^{\hat{\psi}_j(s_i)} (v_i(s) - v_j(s)) f(s) ds_j ds_i.$$

Consider first the case in which ψ_j^* is strictly increasing to the right at \hat{s}_i . Since ψ_j^* is increasing, for any \hat{s}_i, \hat{s}'_i and $\epsilon > 0$, the function:

$$\hat{\psi}_j(s_i) \equiv \begin{cases} \max\{\psi_j^*(s_i) - \epsilon, \psi_j^*(\hat{s}_i)\} & \text{if } s_i \in [\hat{s}_i, \hat{s}'_i] \\ \psi_j^*(s_i) & \text{otherwise,} \end{cases}$$

is also increasing and we can apply the above inequality. After some simple algebraic manipulations, we get:

$$\begin{aligned} & \int_{\hat{s}_i}^{\hat{s}'_i} \int_{\psi_j^*(s_i) - \epsilon}^{\psi_j^*(s_i)} (v_i(s) - v_j(s)) f(s) ds_j ds_i - \\ & \int_{\hat{s}_i}^{\hat{s}'_i} \int_{\psi_j^*(s_i) - \epsilon}^{\max\{\psi_j^*(s_i) - \epsilon, \psi_j^*(\hat{s}_i)\}} (v_i(s) - v_j(s)) f(s) ds_j ds_i \geq 0. \quad (4) \end{aligned}$$

To derive our sufficient condition we divide this inequality by $\epsilon > 0$ and apply to each of the two integrals the following two results respectively:

$$\lim_{\epsilon \rightarrow 0} \frac{\int_{\psi_j^*(s_i) - \epsilon}^{\psi_j^*(s_i)} (v_i(s) - v_j(s)) f(s) ds_j}{\epsilon} = (v_i(s_i, \psi_j^*(s_i)) - v_j(s_i, \psi_j^*(s_i))) f(s_i, \psi_j^*(s_i)),$$

for almost all $s_i \in (\hat{s}_i, \hat{s}'_i)$ and,

$$\lim_{\epsilon \rightarrow 0} \frac{\int_{\psi_j^*(s_i) - \epsilon}^{\max\{\psi_j^*(s_i) - \epsilon, \psi_j^*(\hat{s}_i)\}} (v_i(s) - v_j(s)) f(s) ds_j}{\epsilon} = 0,$$

for any $s_i \in (\hat{s}_i, \hat{s}'_i)$. The first limit can be proved using that for any given s_i , $v_i(s_i, s_j)$ and $v_j(s_i, s_j)$ are both continuous in s_j a.e., a consequence of their monotonicity. The second limit can be proved using that the integrand is bounded and $\lim_{\epsilon \rightarrow 0} \frac{\max\{0, \psi_j^*(\hat{s}_i) - \psi_j^*(s_i) + \epsilon\}}{\epsilon} = 0$ for any $s_i \in (\hat{s}_i, \hat{s}'_i)$ since ψ_j^* is locally strictly increasing to the right at \hat{s}_i .

The case in which ψ_j^* is strictly increasing to the left at \hat{s}_i has a similar analysis but using the function:

$$\hat{\psi}_j(s_i) \equiv \begin{cases} \min\{\psi_j^*(s_i), \psi_j^*(\hat{s}_i) - \epsilon\} & \text{if } s_i < \hat{s}_i \\ \psi_j^*(\hat{s}_i) - \epsilon & s_i \in [\hat{s}_i, \hat{s}'_i] \\ \psi_j^*(s_i) & \text{otherwise,} \end{cases}$$

instead. ■

Claim 2: *The good is allocated according to p^* a.e. when bidders play (b_1^*, b_2^*) . In particular, for any s in the interior of the set of types for which p^* allocates to $i \in \{1, 2\}$, it is verified that $b_i^*(s_i) > b_j^*(s_j)$, $j \neq i$.*

Proof. To prove the claim, it is sufficient to show that for any s that verifies the conditions of the proposition there exists a (\hat{s}_i, \hat{s}_j) and a sequence $\{s_i^n\}$ such that:²²

$$v_i(s_i, \psi_j^*(s_i)) > \lim v_i(s_i^n, \psi_j^*(s_i^n)) \geq \lim v_j(s_i^n, \psi_j^*(s_i^n)) \geq v_j(\hat{s}_i, \hat{s}_j) \geq v_j(\psi_i^*(s_j), s_j) \quad (5)$$

Any s that verifies the conditions of the proposition also verifies that both $(\psi_i^*(s_j), s_j)$ and $(s_i, \psi_j^*(s_i))$ belong to the frontier of the set of types for which p^* allocates to Bidder i . Since the allocation p^* is deterministic and monotone, then $(s_i, \psi_j^*(s_i)) > (\psi_i^*(s_j), s_j)$ and there exists a (\hat{s}_i, \hat{s}_j) such that: (i) it belongs to the frontier of the former set, (ii)

²²We denote $(s_1, s_2) < (\hat{s}_1, \hat{s}_2)$ when $s_1 < \hat{s}_1$ and $s_2 < \hat{s}_2$; and $(s_1, s_2) \leq (\hat{s}_1, \hat{s}_2)$ when $s_1 \leq \hat{s}_1$ and $s_2 \leq \hat{s}_2$ and $(s_1, s_2) \neq (\hat{s}_1, \hat{s}_2)$.

it verifies that $(s_i, \psi_j^*(s_i)) \geq (\hat{s}_i, \hat{s}_j) \geq (\psi_i^*(s_j), s_j)$ and (iii) it is such that ψ_j^* is locally strictly increasing either to the right or the left at \hat{s}_i . (ii) implies the last inequality in Equation (5). (i) and (ii) imply that for any strictly decreasing sequence $\{s_i^n\}$ that starts at s_i and has limit \hat{s}_i , it is verified that $(s_i, \psi_j^*(s_i)) > (s_i^n, \psi_j^*(s_i^n)) \geq (\hat{s}_i, \hat{s}_j)$. This implies the first and the third inequalities in Equation (5). Finally, (iii) and Claim 1 implies the second inequality of Equation (5). \blacksquare

Claim 3: *If f is affiliated, there is a selection of:*

$$\arg \max_{\psi} \int_0^{\psi} (v_2(s_1, s_2) - v_1(s_1, \psi_2^*(s_1))) f(s_1, s_2) ds_1$$

increasing in s_2 .

Proof. To prove the claim, it is sufficient to show that for any $s'_2 > s_2$:

$$\int_{\tilde{\psi}}^{\psi} (v_2(s_1, s_2) - v_1(s_1, \psi_2^*(s_1))) f(s_1, s_2) ds_1 \geq 0, \forall \tilde{\psi} \leq \psi,$$

implies that,

$$\int_{\tilde{\psi}}^{\psi} (v_2(s_1, s'_2) - v_1(s_1, \psi_2^*(s_1))) f(s_1, s'_2) ds_1 \geq 0, \forall \tilde{\psi} \leq \psi.$$

Since v_2 is increasing in s_2 , a sufficient condition (after some straightforward transformations) is that:

$$\int_{\tilde{\psi}}^{\psi} (v_2(s_1, s_2) - v_1(s_1, \psi_2^*(s_1))) f(s_1|s_2) ds_1 \geq 0, \forall \tilde{\psi} \leq \psi,$$

implies that,

$$\int_{\tilde{\psi}}^{\psi} (v_2(s_1, s_2) - v_1(s_1, \psi_2^*(s_1))) f(s_1|s_2) \frac{f(s_1|s'_2)}{f(s_1|s_2)} ds_1 \geq 0, \forall \tilde{\psi} \leq \psi.$$

Since affiliation implies that $\frac{f(s_1|s'_2)}{f(s_1|s_2)}$ is increasing in s_1 , the last condition can be derived from the more general claim that for any $J(s_1)$ increasing and non-negative:

$$\int_{\tilde{\psi}}^{\psi} A(s_1) ds_1 \geq 0, \forall \tilde{\psi} \leq \psi,$$

implies that,

$$\int_{\tilde{\psi}}^{\psi} A(s_1)J(s_1) ds_1 \geq 0, \forall \tilde{\psi} \leq \psi.$$

Integration by parts on the left hand side of the last equation means that it is equal to:

$$J(\tilde{\psi}) \int_{\tilde{\psi}}^{\psi} A(s_1)ds_1 + \int_{\tilde{\psi}}^{\psi} \int_{s_1}^{\psi} A(\tilde{s}_1) d\tilde{s}_1 dJ(s_1),$$

which is non-negative for any $\tilde{\psi} \leq \psi$ as desired, when

$$\int_{\tilde{\psi}}^{\psi} A(s_1) ds_1 \geq 0, \forall \tilde{\psi} \leq \psi.$$

■

Claim 4: (b_1^*, b_2^*) is a Bayesian Nash equilibrium of the English auction.

Proof. Suppose that Bidder 1 plays the proposed strategy and consider Bidder 2's expected payoff when she plays an arbitrary strategy. In this case, Bidder 2 pays a price equal to Bidder 1's bid, i.e. $v_1(s_1, \psi_2^*(s_1))$, if she wins. As in the proof of Proposition 2, the expected payoff of any strategy of Bidder 2 can be characterized with the corresponding function of pivotal types of Bidder 1 $\hat{\psi}_1 : [0, 1] \rightarrow [0, 1]$:

$$\int_0^1 \int_0^{\hat{\psi}_1(s_2)} (v_2(s_1, s_2) - v_1(s_1, \psi_2^*(s_1))) f(s) ds_1 ds_2. \quad (6)$$

Since (b_1^*, b_2^*) induce the allocation p^* a.e. by Claim 2, b_2^* picks $\hat{\psi}_1 = \psi_1^*$ a.e. in the above problem. Thus, to prove that Bidder 2 does not have incentives to deviate, we have to show that $\hat{\psi}_1 = \psi_1^*$ maximizes Equation (6). Claim 3 implies that there is no loss in adding the constraint that $\hat{\psi}_1$ is increasing to this problem.

The integral in Equation (6) is equal to the sum of two integrals:

$$\int_0^1 \int_0^{\hat{\psi}_1(s_2)} (v_2(s_1, s_2) - v_1(s_1, s_2)) f(s) ds_1 ds_2 \quad (7)$$

and,

$$\int_0^1 \int_0^{\hat{\psi}_1(s_2)} (v_1(s_1, s_2) - v_1(s_1, \psi_2^*(s_1))) f(s) ds_1 ds_2. \quad (8)$$

The function ψ_1^* maximizes the first integral subject to $\hat{\psi}_1$ increasing by definition, see Equation (1) for $j = 1$ and $i = 2$. It also maximizes the second integral because the interior of the set $\{(s_1, s_2) : s_2 \geq \psi_2^*(s_1)\}$ is equal to the interior of the set $\{(s_1, s_2) : s_1 \leq \psi_1^*(s_2)\}$. Thus, $\hat{\psi}_1 = \psi_1^*$ maximizes Equation (6) subject to $\hat{\psi}_1$ increasing as desired. ■

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